Stochastic Variational Multi-scale method (FEM Term Project)

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February 26, 2019

Abstract

The Variational Mulitscale formulation for deterministic and stochastic problems in 1D Advection Diffusion and 1D Burgers Equation with dirichlet boundary condition are presented in this report. Linear finite element shape functions are used to represent the physical domain, and spectral basis are used for stochastic domain. The VMS formulation, in general requires a stabilization parameter to be defined which is a function of the advection and diffusion parameter in the equation. The stochasticity of the problem is assumed to be contributed by either an uncertain advection term or uncertain boundary condition. Therefore, in the these cases the stabilization parameters are itself an uncertain term. For the sake of convenience, the stabilization parameters are approximated with their projection in generalized Polynomial chaos (gPC) basis. The unstability of Galerkin methods for advection dominated problem is also highlighted in this work.

Introduction

VMS method was introduced as means for numerical methods to tackle problems with multi-scale phenomenon. The need for such methods was justified by the lack of robustness of Galerkin approach in the presence of multi-scale phenomenon. The crux of method is to decompose the solution variable u to a problem into sum of coarse-scale solution $\bar{u}$ and fine-scale solution u'. We try to solve u' analytically using fine-scale Green's function and subsequently use this to solve $\bar{u}$ numerically. In most cases use of Linear finite element shape functions results in huge simplifications in evaluating the Fine-scale Green's funtion.

In this work, the focus is on implementing VMS formulation [1] to effectively predict the solutions to a problem in stochastic setting. The stochasticity of the problem is handled by spectral representation of the solution in terms of gPC expansion of the random variable [2]. The type of polynomial to be used for gPC expansion is based on the type of distribution of the random variable. This work is limited to uniformly distributed random variables, and hence, the lagrange polynomials are used in gPC expansion. The stabilization parameter which is usually complex trancendental functions of problem parameters are encounterd during the VMS formualtion. A projection based approach is developed to approximate the stabilization parameter as a gPC expansion [3].

The outline of the work is follows – VMS method is explained in the next section. Determinsitic 1D problems are taken in subsequent section and its subsections; their formulations and results to certain cases are presented. Next, formulations and results of stochastic counterparts of the same problems are presented. Finally, the conclusions are elucidated from the results obtained so far in the final conclusion section.

VMS method

Consider the following boundary value problem in space $\Omega \subset \mathbb{R}^d$ where $d \geq 1$ with smooth boundary Γ. Find $u : \Omega \to \mathbb{R}$ such that

$$\mathcal{L}u = f \text{ in } \Omega \tag{1}$$

$$u = g \text{ in } \Gamma \tag{2}$$

where f, g are real functions in Ω and Γ respectively. Assume $\mathcal{L}$ is limited to second order diiferential operator. The weak form of the problem would be:

Let $\delta \subset H^1(\Omega)$ and $\mathcal{V} \subset H^1(\Omega)$ with following properties

$$u = g \quad \forall u \in \delta \tag{3}$$

$$w = 0 \quad \forall v \in \mathcal{V} \tag{4}$$

Find $u \in \delta$ such that $\forall w \in \mathcal{V}$

$$a(w, u) = (w, f) \tag{5}$$

where $(\cdot, \cdot)$ is L_2 inner product in Ω and $a(\cdot, \cdot)$ is bilinear from satisfying $a(w, u) = (w, \mathcal{L}u)$. The VMS solution to this problem can be obrained in the following way. Let

$$u = \bar{u} + u' \quad \forall u \in \delta \tag{6}$$

$$w = \bar{w} + w' \quad \forall v \in \mathcal{V} \tag{7}$$

$\bar{u}$, $\bar{w}$ represent coarse scales while u',w' represent fine scale soultions. Let $\delta = \bar{\delta} \oplus \delta'$ (coarse scale and fine scale trial solution space) and $\mathcal{V} = \bar{\mathcal{V}} \oplus \mathcal{V}'$(coarse and fine scale weighting function space). Assume

$$\bar{u} = g \text{ on } \Gamma \ \forall \bar{u} \in \bar{\delta} \tag{8}$$

$$u' = 0 \text{ on } \Gamma \ \forall u' \in \delta' \tag{9}$$

$$\bar{w} = 0 \text{ on } \Gamma \ \forall \bar{w} \in \bar{\mathcal{V}} \tag{10}$$

$$w' = 0 \text{ on } \Gamma \ \forall \bar{u} \in \mathcal{V}' \tag{11}$$

Assuming the following integration-by-parts holds true $\forall \bar{u}, \bar{w}, u', w'$ belonging to appropriate subspaces

$$a(\bar{w}, u') = (\mathcal{L}^* \bar{w}, u') \tag{12}$$

$$a(w', \bar{u}) = (w', \mathcal{L}u) \tag{13}$$

$$a(w', u') = (w', \mathcal{L}u') \tag{14}$$

Substituting, (6) and (7) onto (5)

$$a(\bar{w} + w', \bar{u} + u') = (\bar{w} + w', f) \quad \forall \bar{w} \in \bar{\mathcal{V}}, \forall w' \in \mathcal{V}' \tag{15}$$

By the virtue of linear independence of $\bar{w}$ and w' the problem can be split into 2:
Problem (1)

$$a(\bar{w}, \bar{u}) + a(\bar{w}, u') = (\bar{w}, f) \quad \forall \bar{w} \in \bar{\mathcal{V}} \tag{16}$$

$$a(\bar{w}, \bar{u}) + (\mathcal{L}^* \bar{w}, u') = (\bar{w}, f) \tag{17}$$

Problem (2)

$$a(w', \bar{u}) + a(w', u') = (w', f) \quad \forall w' \in \mathcal{V}' \tag{18}$$

$$a(w', \mathcal{L}\bar{u}) + (w', \mathcal{L}u') = (w', f) \tag{19}$$

(17) and (19) are obtained by using the (12-14) on (16) and (18). Assuming we solved u' analytically, all that is left to do is to solve (17). We know $\mathcal{L}u' = -(\mathcal{L}\bar{u} - f)$. In order to obtain u' we use Green's function for fine scales.

$$u'(y) = -\int_{\Omega} g'(x,y)(\mathcal{L}\bar{u} - f)d\Omega_x \tag{20}$$

(20) can be rewritten in terms of integral operator M' as $u' = M'(\mathcal{L}\bar{u} - f)$. Substituting in to the (17) we get the VMS weak form:

$$a(\bar{w},\bar{u}) + (\mathcal{L}^*\bar{w}, M'(\mathcal{L}\bar{u})) = (\bar{w}, f) + (\mathcal{L}^*\bar{w}, M'(f)) \quad \forall \bar{w} \in \bar{\mathcal{V}} \tag{21}$$

Under the assumption of linear finite elements and approximations such as $\tau \approx -M'$ and $g'(x,y) \approx g'_e(x,y)$, (21) can be further simplified to its final form

$$a(\bar{w}^h,\bar{u}^h) - \sum_e (\mathcal{L}^*\bar{w}^h, \tau^e \mathcal{L}\bar{u}^h) = (\bar{w}^h, f) - \sum_e (\mathcal{L}^*\bar{w}^h, \tau^e f) \quad \forall \bar{w} \in \bar{\mathcal{V}} \tag{22}$$

where,

$$\tau^e = \frac{1}{meas(\Omega_e)} \int_{\Omega_e}\int_{\Omega_e} g'_e(x,y)d\Omega_x d\Omega_y \tag{23}$$

VMS for Deterministic 1D problems

Advection-Diffusion Equation

For the case of 1D advection-diffusion problem

$$\mathcal{L} = \beta\frac{d}{dx} - \kappa\frac{d^2}{dx^2} \tag{24}$$

and the corresponding adjoint is

$$\mathcal{L}^* = -\beta\frac{d}{dx} - \kappa\frac{d^2}{dx^2} \tag{25}$$

The aim is to solve for $u(x) \in \Omega_x[0,1]$, given $f, \beta, \kappa : \Omega_x \to \mathbb{R}$, g_0 and g_1 with

$$\beta\frac{du}{dx} - \kappa\frac{d^2u}{dx^2} = f(x) \text{ on } \Omega_x \tag{26}$$

with dirichilet boundary conditions

$$u(0) = g_0 \;\&\; u(1) = g_1 \tag{27}$$

The VMS weak form of the problem is

$$\left(\frac{\partial\bar{w}}{\partial x}, \kappa\frac{\partial\bar{u}}{\partial x}\right)_{\Omega_x} + \left(\bar{w}, \beta\frac{\partial\bar{u}}{\partial x}\right)_{\Omega_x} + \sum_e \left(\beta\frac{\partial\bar{w}}{\partial x}, \tau^e\beta\frac{\partial\bar{u}}{\partial x}\right)_{\Omega_x^e} = (\bar{w}, f)_{\Omega_x} + \sum_e \left(\beta\frac{\partial\bar{w}}{\partial x}, \tau^e f\right)_{\Omega_x^e} \tag{28}$$

Before proceeding to solve for the solution we need to compute the elemental stabilization parameter τ^e. For this particular problem τ^e can be computed analytically and can be given as

$$\tau^e = \frac{h_e}{2\beta}\left(coth(Pe_e) - \frac{1}{Pe_e}\right) \tag{29}$$

where $Pe_e = \frac{\beta h_e}{2\kappa}$ is the element Peclet number and h_e is length of element.

The VMS solution can be obtained by the approximation $\bar{u}^h = \hat{u}_A N_A$ where N_As are linear shape functions. Similar approximations is also made for $\bar{w}$ in order to obtain the solution.

Results

For $f(x) = 0$, the following cases were considered. (i) $\frac{\beta L}{2\kappa} = 40$; (ii) $\frac{\beta L}{2\kappa} = 100$. and (iii) $\frac{\beta L}{2\kappa} = 200$ Here, $L = 1$ which is measure of the domain. The equations were then subjected to the boundary conditions $u(0) = 0$ and $u(1) = 1$. The result plots are in figures (1), (2) and (3).

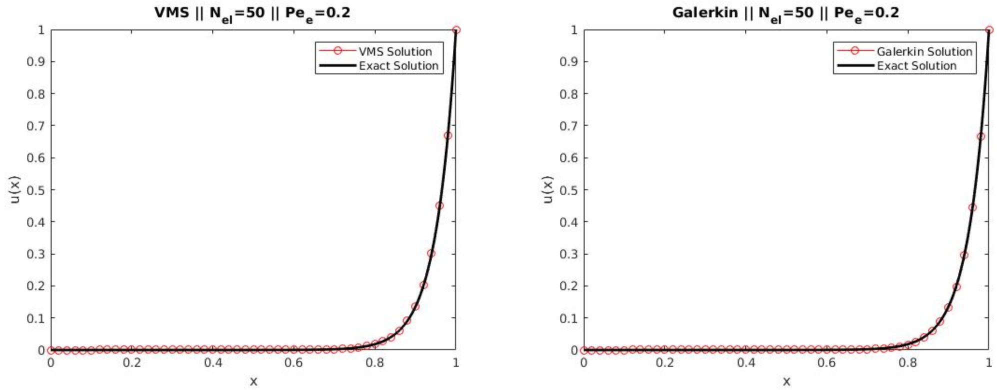


Figure 1: case(i)

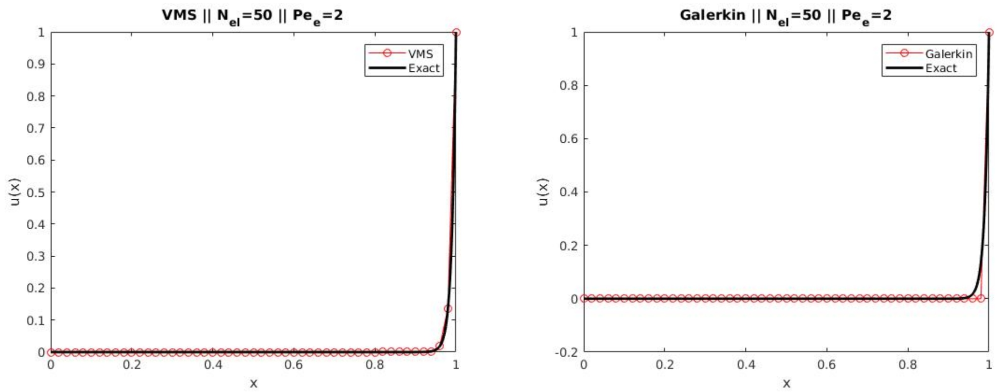


Figure 2: case(ii)

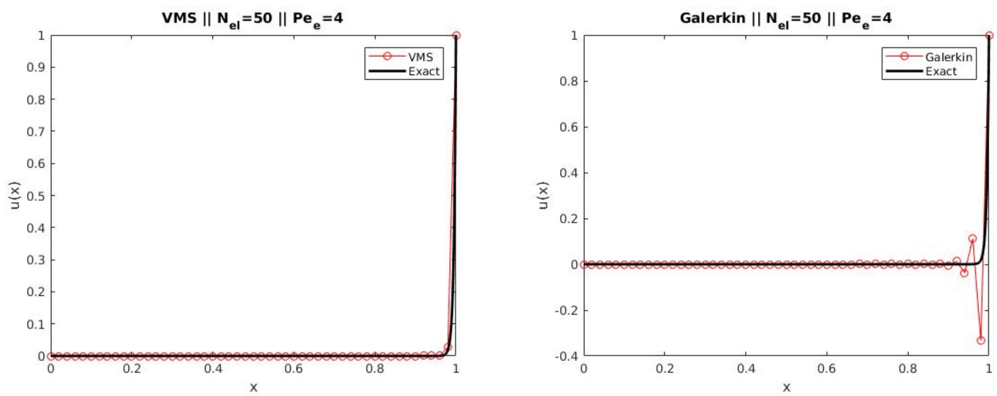


Figure 3: case(iii)

Instabilities in the Galerkin solution with increase in Pe_e can be verified by oscillations observed in the solution.

Burgers Equation

1D burgers equation is very much similar to 1D advection diffusion equation. The only change here is that instead of β we have a u and instaed of κ we have μ. Therefore we have,

$$\mathcal{L}^*(\,) = -\bar{u}\frac{d(\,)}{dx} - \mu\frac{d^2(\,)}{dx^2} \;\; \mathcal{L}(\,) = (\,)\frac{d(\,)}{dx} - \mu\frac{d^2(\,)}{dx^2} \tag{30}$$

The aim is to solve for $u(x) \in \Omega_x[0,1]$, given $f, \mu : \Omega_x \to \mathbb{R}$, g_0 and g_1 with

$$u\frac{du}{dx} - \mu\frac{d^2u}{dx^2} = f(x) \text{ on } \Omega_x \tag{31}$$

with dirichilet boundary conditions

$$u(0) = g_0 \;\&\; u(1) = g_1 \tag{32}$$

The formulation procedure remains similar but we encounter some difficulty due to the non-linear nature of the problem. But this can easily resolved with some valid approxiamtions [4] and the final VMS weak form can be obtained as

$$\left(\frac{\partial \bar{w}}{\partial x}, \mu\frac{\partial \bar{u}}{\partial x}\right)_{\Omega_x} + \left(\bar{w}, \bar{u}\frac{\partial \bar{u}}{\partial x}\right)_{\Omega_x} + \sum_e \left(\bar{u}\frac{\partial \bar{w}}{\partial x}, \tau^e \bar{u}\frac{\partial \bar{u}}{\partial x}\right)_{\Omega_x^e} = (\bar{w}, f)_{\Omega_x} + \sum_e \left(\bar{u}\frac{\partial \bar{w}}{\partial x}, \tau^e f\right)_{\Omega_x^e} \tag{33}$$

Once again τ^e computed analytically and can be given by

$$\tau^e = \frac{h_e}{2\bar{u}^e}\left(coth(Pe_e) - \frac{1}{Pe_e}\right) \tag{34}$$

where $Pe_e = \frac{\bar{u}^e h_e}{2\mu}$ is the element Peclet number and h_e is length of element. Rewriting everything in terms of linear shape functions basis $\bar{w} = \hat{C}_A N_A$ and $\bar{u} = \hat{D}_B N_B$. We arrive at $\hat{C}_A G_A = 0$ where

$$\begin{aligned} G_A(\hat{\boldsymbol{D}}) = {} & \left(N_A, N_B N_{C,x}\right)\hat{D}_B\hat{D}_C + \left(N_{A,x}, \mu N_{B,x}\right)\hat{D}_B \\ & + \sum_e \left(N_B N_{A,x}, \tau^e N_C N_{D,x}\right)_{\Omega_e}\hat{D}_B\hat{D}_C\hat{D}_D \\ & - (N_A, f) - \sum_e \left(N_{A,x} N_B, \tau^e f\right)_{\Omega_e}\hat{D}_B \end{aligned} \tag{35}$$

Implementing Newton-Raphson for the k^{th} iteration

$$\begin{aligned} R_I &= G_I(\hat{\boldsymbol{D}}^k) \quad \text{...Residual} \\ J_{IJ} &= \left[\frac{dG_I(\hat{\boldsymbol{D}}^k)}{dD_J^k}\right] \quad \text{...Jacobian} \end{aligned} \tag{36}$$

$$\hat{\boldsymbol{D}}^{k+1} = \hat{\boldsymbol{D}}^k - [\boldsymbol{J}]^{-1}\{\boldsymbol{R}\} \tag{37}$$

Computing element level terms,

$$
\begin{aligned}
R_i^e &= \left(N_i, \bar{u}^e \frac{d\bar{u}^e}{dx}\right) + \left(N_{i,x}, \mu \frac{d\bar{u}^e}{dx}\right) + \left(\bar{u}^e N_{i,x}, \tau^e \bar{u}^e \frac{d\bar{u}^e}{dx}\right) - (N_i, f^e) - (\bar{u}^e N_{i,x}, \tau^e f^e) \\
J_{ij}^e &= \left(N_i, \bar{u}^e N_{j,x}\right) + \left(N_i, N_j \frac{d\bar{u}^e}{dx}\right) + \left(N_{i,x}, \mu N_{j,x}\right) \\
&\quad + \left(\bar{u}^e N_{i,x}, \tau^e \bar{u}^e N_{j,x}\right) + \left(\bar{u}^e N_{i,x}, \tau^e N_j \frac{d\bar{u}^e}{dx}\right) + \left(N_j N_{i,x}, \tau^e \bar{u}^e \frac{d\bar{u}^e}{dx}\right) - \left(N_j N_{i,x}, \tau^e f^e\right)
\end{aligned}
\tag{38}
$$

where,

$$\bar{u}^e = \sum_{i=1}^{nshl} \hat{D}_i N_i \tag{39}$$

$$\frac{d\bar{u}^e}{dx} = \sum_{i=1}^{nshl} \hat{D}_i N_{i,x} \tag{40}$$

$$f^e = \sum_{i=1}^{nshl} \hat{f}_i N_i \tag{41}$$

Results

The analytical solution to burger's equation can only be found for $f(x) = 0$. Even then, we end up with a transcendental equation based on the boundary condition which has infinitely many solution. The use of Newton-Raphson in such a case is not effective. So in this study, manufactured solutions are first taken, and then the corresponding $f(x)$ is fed into the code to obtain the solution. One such case is presented in this report

$$u_{exact}(x) = 1 + x - \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1} \implies \quad u(0) = 1 \;\; u(1) = 1$$

Some results are plotted in the figures (4)-(7).

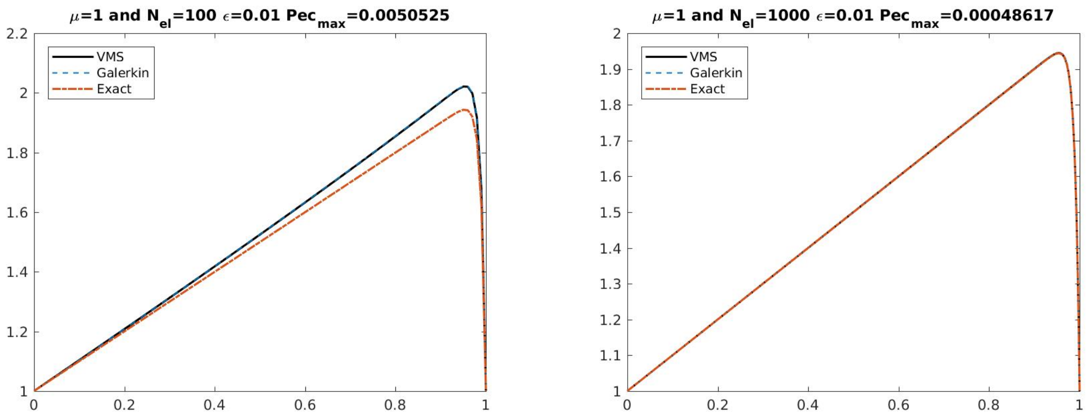


Figure 4: case(i)

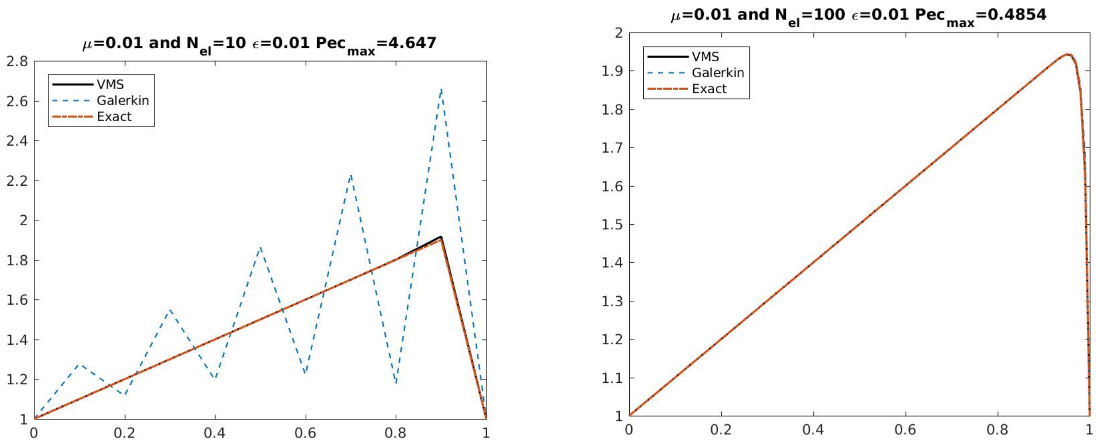


Figure 5: case(ii)

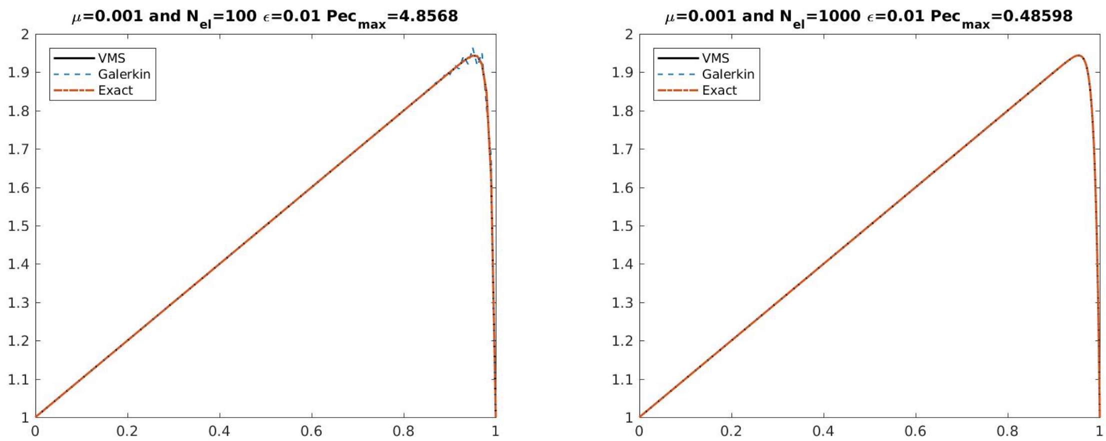


Figure 6: case(iii)

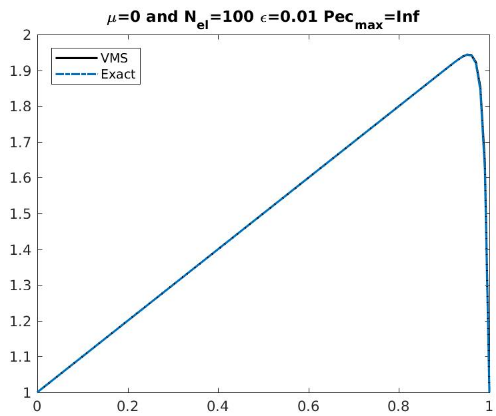


Figure 7: case(iv) Galerkin solution doesn't converge for when $\mu = 0$

Once again, similar observations regarding the oscillations in Galerkin solution can be made for when the element Peclet number reaches a considerable value.

VMS for Stochastic 1D problems

In stochastic setting the trial and weighting solution spaces has to be redefined. The solution spaces for the variational problem is obtained through a tensor product of suitable trial spaces over the physical and stochastic domains.

$$\delta = \{u \in \delta_x \otimes \delta_y; u = g \text{ on } \Gamma_x\} \tag{42}$$

$$\mathcal{V} = \{w \in \mathcal{V}_x \otimes \mathcal{V}_y; w = 0 \text{ on } \Gamma_x\} \tag{43}$$

where δ_x and $\mathcal{V}_x$ are similar to out previous definitions. But $\mathcal{V}_y, \delta_y = L_2(\Omega_y)$, where Ω_y is the stochastic domain of the problem. The new stochastic VMS weak form will be

$$\mathbb{E}\left[a(\bar{w}^h, \bar{u}^h)_{\Omega_x} - \sum_e (\mathcal{L}^* \bar{w}^h, \tau^e \mathcal{L}\bar{u}^h)_{\Omega_x^e}\right] = \mathbb{E}\left[(\bar{w}^h, f)_{\Omega_x} - \sum_e (\mathcal{L}^* \bar{w}^h, \tau^e f)_{\Omega_x^e}\right] \quad \forall \bar{w} \in \bar{\mathcal{V}} \tag{44}$$

where $\mathbb{E}$ is expectation operator defined by

$$\mathbb{E}[\cdot] = \int_{\Omega_y} (\cdot)\rho(y) d\Omega_y \tag{45}$$

$\rho(y)$ is probability distribution fucntion of random vector y. While any number of random variables can influence a physical system, this report only focuses on problems with one uniformly distributed random variable. Therefore $\bar{u}^h$ and $\bar{w}^h$ are approximated as tensor product of Linear Finite elements for physical space Ω_x and spectral basis for stochastic domain Ω_y

$$\bar{w}^h = \sum_A \sum_{I=1}^{M} \hat{w}_A^I N_A(x)\phi_I(y) \quad \bar{u}^h = \sum_B \sum_{J=1}^{M} \hat{u}_B^J N_B(x)\phi_J(y) \tag{46}$$

where

$$M = dim(\bar{\mathcal{V}}_y)\frac{(q+p)!}{q!p!} \tag{47}$$

Here q is the number of random variable (q=1, througout this report) and p is the atmost order of polynomial used.

Advection-Diffusion Equation with uncertain Advection term

Solve for $u(x, y) \in [\Omega_x \times \Omega_y] \rightarrow \mathbb{R}$

$$\beta(y)\frac{\partial u}{\partial x} - \kappa(x)\frac{\partial^2 u}{\partial x^2} = f \quad \Omega^x \in [0, 1] \tag{48}$$

with dirichilet boundary conditions

$$u(0, y) = g_0 \;\&\; u(1, y) = g_1 \tag{49}$$

where, $\kappa(x)$ is deterministic and $\beta(y)$ is uncertain function of random variable in $y \in \mathcal{U}[-1, 1]$. The VMS weak form is given by

$$B(\bar{w}, \bar{u}) = (\bar{w}, f) \tag{50}$$

where

$$B(\bar{w},\bar{u}) = \mathbb{E}\left[\left(\frac{\partial \bar{w}}{\partial x},\kappa\frac{\partial \bar{u}}{\partial x}\right)_{\Omega_x} + \left(\bar{w},\beta(y)\frac{\partial \bar{u}}{\partial x}\right)_{\Omega_x} + \sum_e\left(\beta(y)\frac{\partial \bar{w}}{\partial x},\tau^e(y)\beta(y)\frac{\partial \bar{u}}{\partial x}\right)_{\Omega_x^e}\right] \tag{51}$$

$$(\bar{w},f) = \mathbb{E}\left[(\bar{w},f)_{\Omega_x} + \sum_e\left(\beta(y)\frac{\partial \bar{w}}{\partial x},\tau^e(y)f\right)_{\Omega_x^e}\right] \tag{52}$$

The stabilization parameter is now given by

$$\tau^e(y) = \frac{h_e}{2\beta(y)}\left(coth(Pe_e(y)) - \frac{1}{Pe_e(y)}\right) \tag{53}$$

$$Pe_e(y) = \frac{|\beta(y)|h_e}{2\kappa} \tag{54}$$

The stabilization parameter is a complicated transcendental function of the random variables. Therefore, the next objective is to approximate the stabilization parameter in spectral basis. Doing this will reduce the complexity of the integrals by huge extent. Firstly, we use the doubly-asymptotic approximation [5] to span the advective-diffusive limit

$$\tau^e(y) \approx \tilde{\tau}^e(y) = ((\tau^{e,adv}(y))^-2 + 9(\tau^{e,diff})^-2)^{-1/2} \tag{55}$$

$$\tau^{e,adv}(y) = \frac{h_e}{2|\beta(y)|} \tag{56}$$

$$\tau^{e,diff} = \frac{h_e^2}{4\kappa} \tag{57}$$

The approxiated $\tilde{\tau}^e(y)$ is then projected onto a spectral basis to obtain a gPC approximation of stabilization parameter $\tilde{\tilde{\tau}}^e(y)$. The details of the approxiamtion are well elucidated in [3].

To obtain the VMS solution Linear Finite elements for physical space Ω_x and spectral basis for stochastic domain Ω_y are used,

$$\bar{w} = \sum_{A,I}\hat{w}_A^I N_A(x)\phi_I(y) \quad \bar{u} = \sum_{B,J}\hat{u}_B^J N_B(x)\phi_J(y) \tag{58}$$

Substituting onto VMS weak form,

$$\begin{aligned}&\mathbb{E}\left[(N_{A,x}\phi_I(y),\kappa N_{B,x}\phi_J(y))_{\Omega_x} + (N_A\phi_I(y),\beta(y)N_{B,x}\phi_J(y))_{\Omega_x}\right]\hat{u}_B^J + \\ &\mathbb{E}\left[\sum_e(\beta(y)N_{A,x}\phi_I(y),\tau^e(y)\beta(y)N_{B,x}\phi_J(y))_{\Omega_x}\right]\hat{u}_B^J \\ &= \mathbb{E}\left[(N_A\phi_I(y),f)_{\Omega_x} + \sum_e(\beta(y)N_{A,x}\phi_I(y),\tau^e(y)f)_{\Omega_x}\right]\end{aligned} \tag{59}$$

which in matrix form yields

$$\left[K_{AB}^{IJ}\right]\{\hat{u}_B^J\} = \{F_A^I\} \tag{60}$$

where,

$$F_A^I = (N_A,f)\mathbb{E}[\phi_I(y)] + \sum_e(N_{A,x},f)\mathbb{E}[\beta(y)\phi_I(y)\tau^e(y)] \tag{61}$$

$$K_{AB}^{IJ} = (N_{A,x}, \kappa N_{B,x})\mathbb{E}[\phi_I(y)\phi_J(y)] + (N_A, N_{B,x})\mathbb{E}[\phi_I(y)\beta(y)\phi_J(y)] + \sum_e (N_{A,x}, N_{B,x})\mathbb{E}[\beta(y)\phi_I(y)\tau^e(y)\beta(y)\phi_J(y)] \tag{62}$$

The following approximations can be made for $\beta(y)$, $\tau(y)$ and their products encountered in the expectation calculations.

$$\beta(y) \approx \tilde{\beta}(y) = \sum_{K=1}^{M} \hat{\beta}_K \phi_K(y) \tag{63}$$

$$\tau(y) \approx \tilde{\tau}(y) \approx \tilde{\tilde{\tau}}(y) = \sum_{K=1}^{M} \hat{\tau}_K \phi_K(y) \tag{64}$$

$$\beta(y)\tau(y) \approx \gamma(y) = \sum_{K=1}^{M} \hat{\gamma}_K \phi_K(y) \tag{65}$$

$$\beta(y)\tau(y)\beta(y) \approx \gamma(y)\beta(y) \approx \eta(y) = \sum_{K=1}^{M} \hat{\eta}_K \phi_K(y) \tag{66}$$

The coefficients $\hat{\gamma}$ and $\hat{\eta}$ can be calculated same way as $\hat{\tau}$. Therefore,

$$\mathbb{E}[\phi_I(y)\beta(y)\phi_J(y)] \approx \sum_{K=1}^{M} \hat{\beta}_K \mathbb{E}[\phi_I(y)\phi_K(y)\phi_J(y)] \tag{67}$$

$$\mathbb{E}[\beta(y)\phi_I(y)\tau^e(y)] \approx \sum_{J=1}^{M}\sum_{K=1}^{M} \hat{\beta}_K \hat{\tau}_J \mathbb{E}[\phi_I(y)\phi_J(y)\phi_K(y)] \tag{68}$$

$$\mathbb{E}[\beta(y)\phi_I(y)\tau^e(y)\beta(y)\phi_J(y)] \approx \sum_{K=1}^{M} \hat{\eta}_K \mathbb{E}[\phi_I(y)\phi_K(y)\phi_J(y)] \tag{69}$$

All the above expectation terms can be calculated analytically. The boundary conditions are then implemented in the following way

$$u(0,y) = u(x_1,y) = \sum_{B,J} \hat{u}_B^J N_B(x_1)\phi_J(y) = \sum_J \hat{u}_1^J \phi_J(y) \quad \text{since } N_A(x_i) = \delta_{A,i} \tag{70}$$

$$u(1,y) = u(x_{nnp},y) = \sum_{B,J} \hat{u}_B^J N_B(x_{nnp})\phi_J(y) = \sum_J \hat{u}_{nnp}^J \phi_J(y) \tag{71}$$

Using spectral projection and the orthonormal property of the polynomials,

$$g_1 \int_{-1}^{1} \rho(y)\phi_I(y)dy = \sum_J \hat{u}_1^J \int_{-1}^{1} \rho(y)\phi_I(y)\phi_J(y)dy = \hat{u}_1^I \quad \forall I \tag{72}$$

$$g_2 \int_{-1}^{1} \rho(y)\phi_I(y)dy = \sum_J \hat{u}_{nnp}^J \int_{-1}^{1} \rho(y)\phi_I(y)\phi_J(y)dy = \hat{u}_{nnp}^I \quad \forall I \tag{73}$$

Results

The uncertain advection term is considered to be of the form $\beta(y) = 1 + y^2$ where $y \in \mathcal{U}[-1,1]$. The analytical solution can be found for this equation when $f(x) = 1$.

$$u_{exact}(x,y) = -\frac{xe^{\frac{\beta(y)}{\kappa}} + e^{x\frac{\beta(y)}{\kappa}} + x - 1}{\beta(y)(e^{\frac{\beta(y)}{\kappa}} - 1)}$$

Results for $\kappa = 0.001$ for $P = 2, 4$ are are presented in figures (8) and (9).

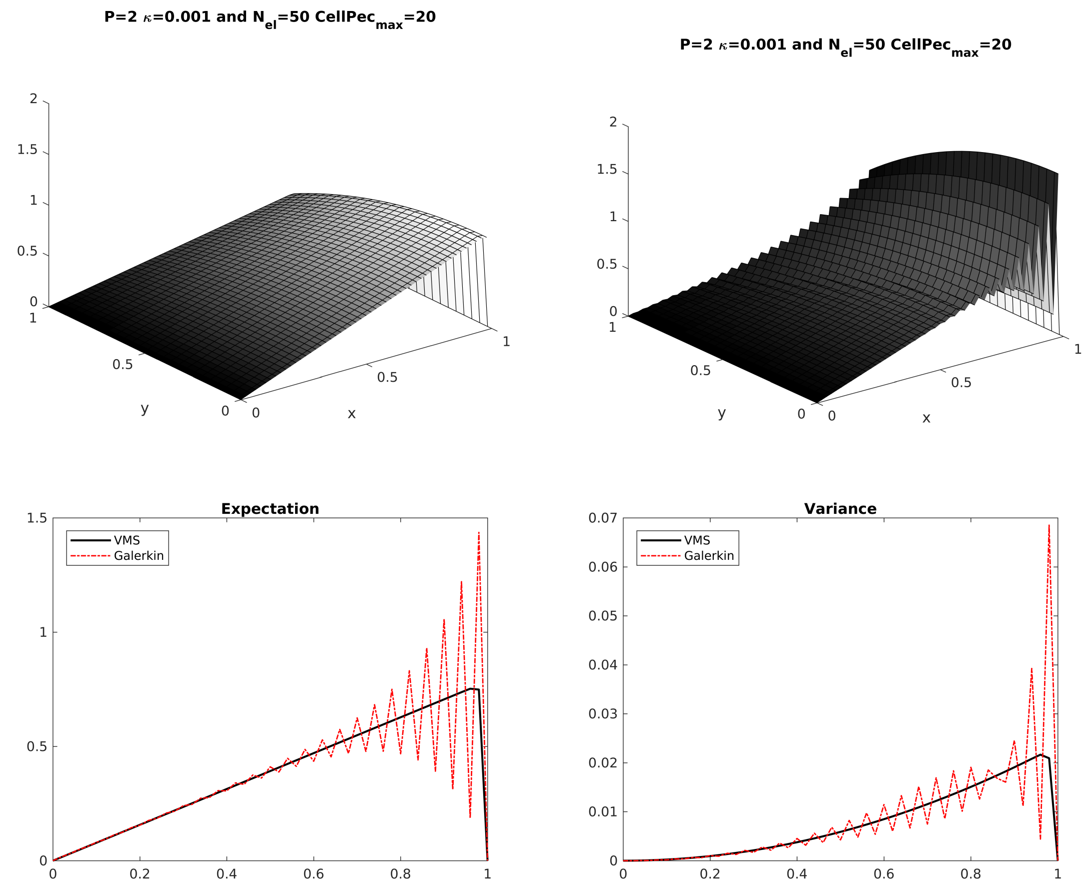
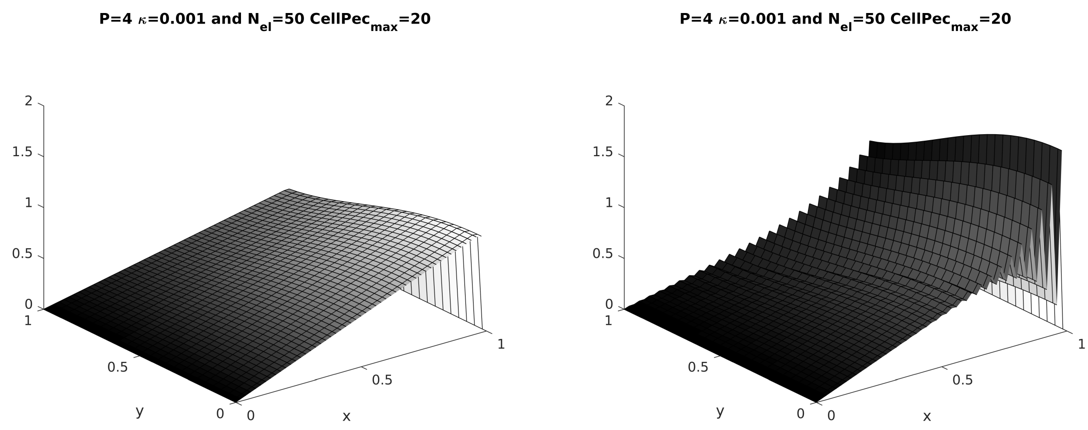


Figure 8: case(i) $P = 2$



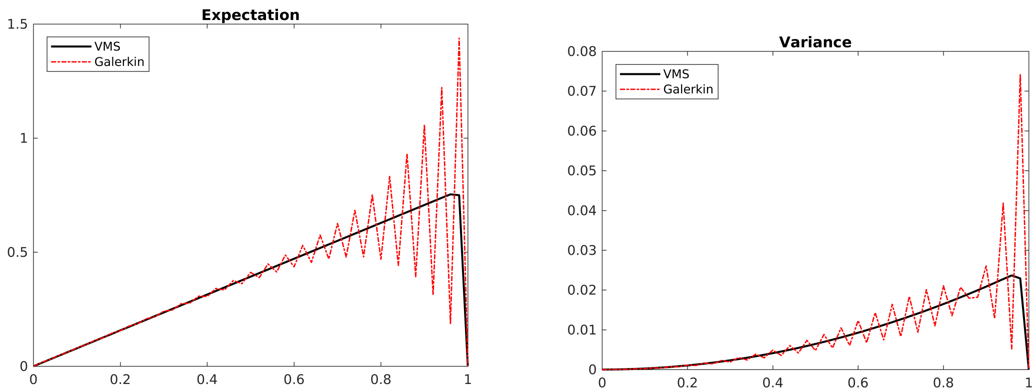


Figure 9: case(ii) $P = 4$

The accuracy of VMS solution has improved when the maximum polynomial order P is increased. This is mostly because the spectral projections of the variables eqns(63-66) are truncated at higher order which is a better approximation than the former. However, increasing the polynomial order will result in additional computational costs.

Burgers Equation with uncertain boundary term

Solve for $u(x, y) \in [\Omega_x \times \Omega_y] \to \mathbb{R}$

$$u\frac{\partial u}{\partial x} - \mu\frac{\partial^2 u}{\partial x^2} = f \quad \Omega^x \in [0, 1] \tag{74}$$

with dirichilet boundary conditions

$$u(0, y) = g_0 \text{ (deterministic bc) } \; \& \; u(1, y) = g_1(y) \text{ (uncertain bc)} \tag{75}$$

The approximate VMS weak form copupled with linear FE shape function assumption yields the following equation

$$\mathbb{E}\left[\left(\bar{w}, \bar{u}\frac{d\bar{u}}{dx}\right)_{\Omega^x} + \left(\frac{d\bar{w}}{dx}, \mu\frac{d\bar{u}}{dx}\right)_{\Omega^x} + \sum_e\left(-\bar{u}\frac{d\bar{w}}{dx}, -\tau^e(y)\bar{u}\frac{d\bar{u}}{dx}\right)_{\Omega^x_e}\right]$$
$$= \mathbb{E}\left[(\bar{w}, f)_{\Omega^x} + \sum_e\left(-\bar{u}\frac{d\bar{w}}{dx}, -\tau^e(y) f\right)_{\Omega^x_e}\right] \tag{76}$$

where,

$$\tau^e(y) \approx \tilde{\tau}^e(y) = ((\tau^{e,adv}(y))^{-}2 + 9(\tau^{e,diff})^{-}2)^{-1/2} \tag{77}$$

$$\tau^{e,adv}(y) = \frac{h_e}{2|\bar{u}^e(y)|} \tag{78}$$

$$\tau^{e,diff} = \frac{h_e^2}{4\mu} \tag{79}$$

Once again we need to project $\tilde{\tau}^e(y)$ onto spectral basis to obtain the final approximate stabilization parameter $\tilde{\tilde{\tau}}^e(y)$. The following approximations can be made for u^e, $\frac{du^e}{dx}$, τ^e

and their products

$$\bar{u}^e = \sum_{J=1}^{M}\left(\sum_{b=1}^{nshl}\hat{u^e}_b^J N_b^e(x)\right)\phi_J(y) = \hat{\alpha}_J(x)\phi_J(y) \tag{80}$$

$$\frac{d\bar{u}^e}{dx} = \sum_{J=1}^{M}\left(\sum_{b=1}^{nshl}\hat{u^e}_b^J N_{b,x}^e(x)\right)\phi_J(y) = \hat{\beta}_J(x)\phi_J(y) \tag{81}$$

$$\tau(y) \approx \tilde{\tau}(y) \approx \tilde{\tilde{\tau}}(y) = \hat{\tau}_J(x)\phi_J(y) \tag{82}$$

$$\bar{u}^e\tau^e = \hat{\theta}_J(x)\phi_J(y) \tag{83}$$

$$\bar{u}^e\tau^e\frac{d\bar{u}^e}{dx} = \hat{\gamma}_J(x)\phi_J(y) \tag{84}$$

$$\bar{u}^e\tau^e\bar{u}^e = \hat{\delta}_J(x)\phi_J(y) \tag{85}$$

All coefficients in the above equations are function of x and are to be evaluated at a particular quadrature points while peroming the integrals in physical domain.

Implementing Newton Raphson and computing element level terms,

$$\begin{aligned}\{R_a^e\}^i = &\left(N_a, \hat{\alpha}_j\hat{\beta}_k\mathbb{E}[\phi_i\phi_j\phi_k]\right)_{\Omega_x} + \left(N_{a,x}, \mu\hat{\beta}_j\mathbb{E}[\phi_i\phi_j]\right)_{\Omega_x} + \left(N_{a,x}, \hat{\beta}_j\hat{\delta}_k\mathbb{E}[\phi_i\phi_j\phi_k]\right)_{\Omega_x} \\ &- \left(N_a, f^e\mathbb{E}[\phi_i]\right)_{\Omega_x} - \left(N_{a,x}, f^e\hat{\alpha}_j\hat{\tau}_k\mathbb{E}[\phi_i\phi_j\phi_k]\right)_{\Omega_x}\end{aligned} \tag{86}$$

$$\begin{aligned}[J_{ab}^e]^{ij} = &\left(N_a, N_{b,x}\hat{\alpha}_k\mathbb{E}[\phi_i\phi_j\phi_k]\right)_{\Omega_x} + \left(N_a, N_b\hat{\beta}_k\mathbb{E}[\phi_i\phi_j\phi_k]\right)_{\Omega_x} + \left(N_{a,x}, \mu N_{b,x}\mathbb{E}[\phi_i\phi_j]\right)_{\Omega_x} \\ &+ \left(N_{a,x}, N_{b,x}\hat{\delta}_k\mathbb{E}[\phi_i\phi_j\phi_k]\right)_{\Omega_x} + \left(N_{a,x}, N_b\hat{\gamma}_k\mathbb{E}[\phi_i\phi_j\phi_k]\right)_{\Omega_x} + \left(N_bN_{a,x}, \hat{\gamma}_k\mathbb{E}[\phi_i\phi_j\phi_k]\right)_{\Omega_x} \\ &- \left(N_bN_{a,x}, f^e\hat{\tau}_k\mathbb{E}[\phi_i\phi_j\phi_k]\right)_{\Omega_x}\end{aligned} \tag{87}$$

$$\Delta U_B^I = [Jac]_{AB}^{IJ} \setminus \{R\}_A^I \tag{88}$$

$$U_{new} = U_{old} - \Delta U \tag{89}$$

Results

For the case when $f(x) = 1$ and the boundary conditions given by $u(0, y) = 0$ and $u(1, y) = y$. Since, the analytical solution cannot be found, efficiency of the algorithm is tested against individual realizations of boundary conditions to previously-verified deterministic VMS code. The results for some selective cases are shown below.

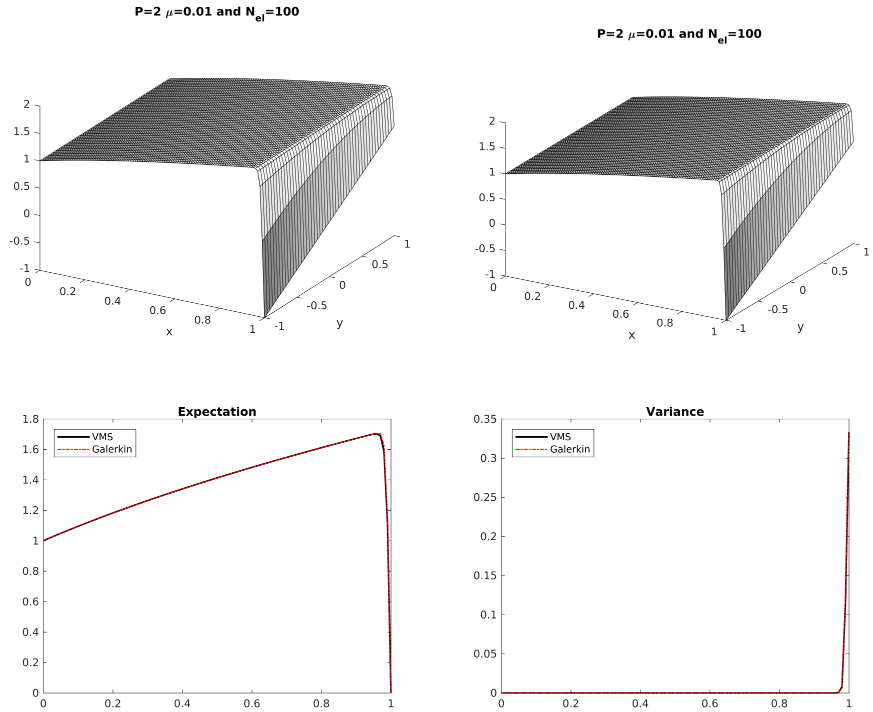


Figure 10: case(i) $\mu = 0.01$ $P = 2$

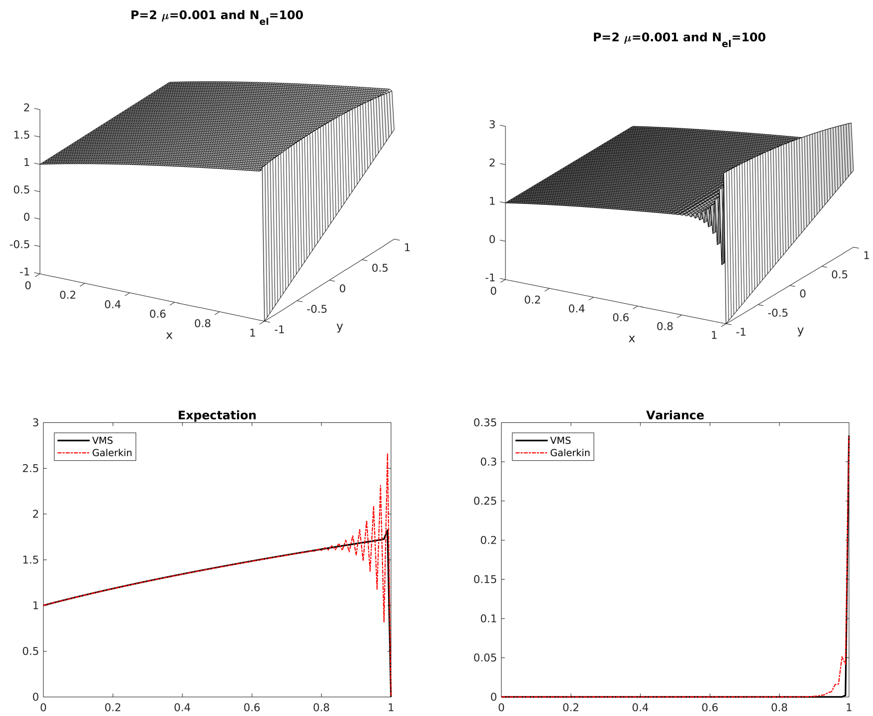


Figure 11: case(ii) $\mu = 0.001$ $P = 2$

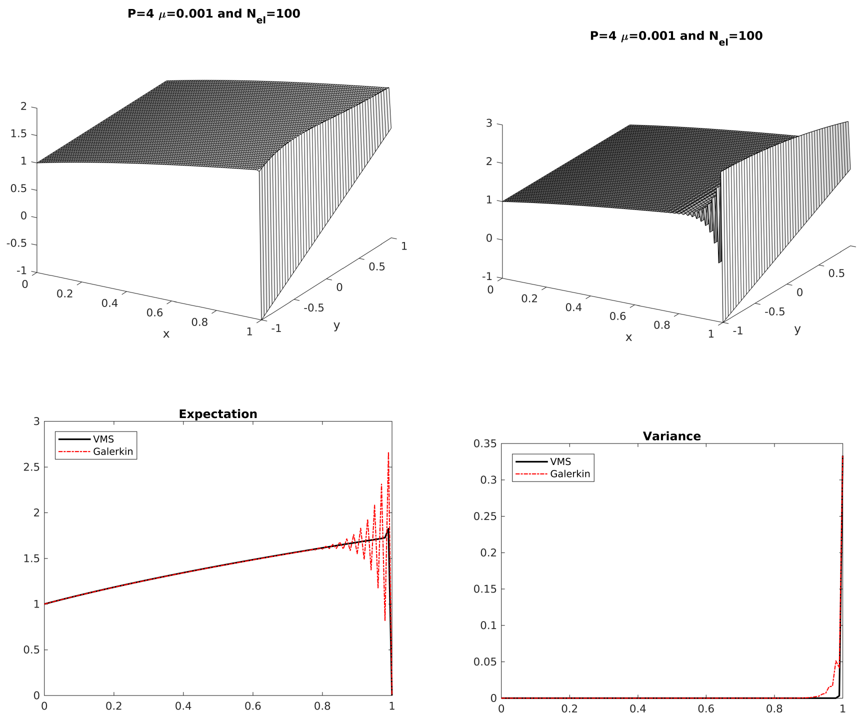


Figure 12: case(iii) $\mu = 0.001$ $P = 4$

Conclusions

In order to perform detailed study of Stochastic VMS [3], [1] and [2] were throughly surveyed. [1] explains a fundamental approach for VMS method and its formulation, while [2] focuses on numerical methods to predict a stochastic system.

In this work, VMS formulations for deterministic and stochastic 1D problems are presented. The results obtained presents a solid case for using VMS method over conventional Galerkin method. Especially in cases where the element Peclet number reaches a considerable value, the Galerkin method proves to be unstable. VMS tackles this with use of stabilization parameter in its formulation. To incorporate the stabilization parameter into the numerical method, a projection based approach is used which approximates the parameter in spectral basis. The solutions obtained through VMS can also be improved with increasing the polynomial order used, which comes at a computational cost. Further investigations is deemed necessary for Stochastic Burger's equation since for certain cases the solution is not converging efficiently.

All the Finite Element codes used here were developed in MATLAB. They can be found in the SCOREC's file scratch lore in the following folder: `\lore\vittav\FEM-Project`

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